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EQUATIONALLY DEFINABLE FUNCTORS AND POLYNOMIAL MAPPINGS

Andrzej PRÓSZYŃSKI
Institute of Mathematics, WSP Bydgoszcz, Poland

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We look for relations satisfied by mappings obtained from forms of degree m. For this goal, we develop a general theory of equationally definable functors. We also give a connection between the type of relations and the structure of the functor $\bar{\Gamma}^m$ defined elsewhere.

Introduction

Let R be a commutative ring. Any form $F\in R[T_1,\dots,T_n]_m$ determines a polynomial mapping $f:R^n\to R$. More generally, we can consider mappings $f:M\to N$ between R-modules obtained from forms in the sense of Roby $F\in\mathscr{P}^m(M,N)$ (see [9] for definition). These mappings constitute an R-module denoted by $\mathrm{Hom}_R^m(M,N)$, and, in the natural way, Hom_R^m is a functor from R-$\mathbf{Mod}^0\times R$-$\mathbf{Mod}$ to R-$\mathbf{Mod}$.

It is known (see, for instance, [3]) that any mapping $f\in\mathrm{Hom}_R^m(M,N)$ satisfies the following relations:

(A1) $f(rx)=r^m f(x)$ for $r\in R$ and $x\in M$,

(A2) $\Delta^m f:M^m\to N$ is m-linear, where

$$(\Delta^m f)(x_1,\dots,x_m)=\sum_{H\subset[1,m]}(-1)^{m-|H|}f\Big(\sum_{i\in H}x_i\Big).$$

A mapping f satisfying (A1) and (A2) is called an *m-application* (see [3]), and the resulting condition $\Delta^{m+1}f=0$ means (in the terms of [4]) that any m-application is a polynomial map of degree $\le m$. In the natural way, we obtain the functor of m-applications $\mathrm{Appl}_R^m: R$-$\mathbf{Mod}^0\times R$-$\mathbf{Mod}\to R$-$\mathbf{Mod}$. It is known that $\mathrm{Hom}_R^m\ne\mathrm{Appl}_R^m$ in general, therefore another relations for Hom^m can hold. Our problem is to find these relations, and, first of all, answer the question whether Hom^m is given by a system of relations, as it occurs in the case of Appl^m. In any case, we can ask what are 'all possible' relations for Hom^m.

We are looking for relations of the form $\sum_j r_j f(\sum_k s_{jk}x_k)=0$, where r_j, s_{jk} ($j,k=1,2,\dots$) denote fixed elements of R, almost all equal to zero, and x_k are assumed to be arbitrary elements of the module. A functor constituted by mappings satisfying a system of such relations will be called equationally definable (or, shortly, an ED-functor). Such is, for instance, the functor Appl^m.

It is proved in Section 1 that any ED-functor is representable. Unfortunately, this is not the case with Hom^m (see [7] or [5]). However, it is proved in Section 2 that there exists a smallest ED-functor, $\mathrm{ED}(\mathrm{Hom}^m)$, containing Hom^m, and, moreover, this functor is effectively computable (Corollary 2.5). The remaining problem is to find relations defining $\mathrm{ED}(\mathrm{Hom}^m)$, or, equivalently, all relations satisfied by Hom^m. This question is open in general, but it can be reduced to the problem of finding generators for the kernel of the natural transformation h^m between the functor Δ^m representing Appl^m and the functor Γ^m of mth divided power. In this version, the problem was investigated (and partially solved) in [7] and [8]. As a corollary, we obtain the complete solution for $m \leq 3$ (Theorem 3.6).

It is proved in Section 3 that any system of relations in question is equivalent to a system of relations of the following form:

$$\sum_j r_j(\Delta^n f)(s_{j1}x_1, \ldots, s_{jn}x_n) = 0$$

for $n = 0, 1, \ldots$ This system is not independent in the sense that relations for one number n can give us relations for other numbers. A subsystem of relations is called n-covering if it provides 'all possible' relations for the fixed n. For example, Theorem 3.6 presents a system of n-covering relations of Hom^m for $n \leq 1$ and $n \geq m-1$, generalizing the result mentioned above for $m \leq 3$.

Although n-covering relations of Hom^m are not known for $1 < n < m-1$, the shape of these relations can be examined. An n-covering system is called strong (see Section 4) if it consists of relations of the following form:

$$\sum_j r_j(a_1, a_2, \ldots) f\Big(\sum_k s_{jk}(a_1, a_2, \ldots) x_k\Big) = 0,$$

where $a_1, a_2, \ldots$ denote arbitrary elements of R and r_j, s_{jk} are fixed polynomials over $\mathbb{Z}$ (more generally, over a commutative ring S). In this case, the functor in question is assumed to be defined (as the functor Hom^m) over all rings (respectively, over all S-algebras). The Main Theorem 6.2 confirms that Hom^m admits a strong system of n-covering relations iff $n \leq 1$ or $m \leq 5$ or $n \geq m-1$ (more generally, iff $n \leq 1$ or $m < 2(d+1)$ or $m - n < 2(d-1)$, where $d = \min\{|S/M| : M \in \mathrm{Max}(S)\}$). Therefore the strong relations remain unknown merely for (m, n) equal to (4, 2), (5, 2) and (5, 3). (Cf. the ending of Section 4.)

Before proving the Main Theorem, it is necessary to have more detailed information about the functor $\tilde{\Gamma}^m$ investigated in [5], [6] and [7]. This is the aim of Section 5.

In the paper all rings are assumed to be commutative with 1. If it is not necessary, the symbol of the ring R (as in Hom_R^m, etc.) will be omitted. The elements of the standard basis of $R^n = \bigoplus_{i=1}^n R$ (possible $n = \infty$) will be denoted by $e_1, e_2, \ldots$.

1. ED-functors

In this section, we consider *map-functors*, i.e. subfunctors of the functor Map : R-$\mathbf{Mod}^0 \times R$-$\mathbf{Mod} \to R$-$\mathbf{Mod}$ of all mappings between modules over a fixed ring R. The functor Map is represented by $F: R$-$\mathbf{Mod} \to R$-$\mathbf{Mod}$, where $F(M)$ denotes the free R-module on the set M. Any map-functor A gives us a subfunctor K_A of F defined by:

$$K_A(M) = \{x \in F(M): \forall_{N \in R\text{-}\mathbf{Mod}} \forall_{f \in A(M,N)} \bar{f}(x) = 0\},$$

where $\bar{f}$ denotes the unique homomorphism on $F(M)$ extending f, $\bar{f}([x]) = f(x)$. Conversely, for any subfunctor K of F we introduce a map-functor A_K defined by

$$A_K(M, N) = \{f \in \mathrm{Map}(M, N): \bar{f}(K(M)) = 0\}.$$

It is clear that $A_K(M, N) \simeq \mathrm{Hom}(F(M)/K(M), N)$ and hence A_K is represented by F/K.

Lemma 1.1. *Operations $A \mapsto K_A$ and $K \mapsto A_K$ introduced above define a one-to-one Galois-type correspondence between all representable map-functors and all subfunctors of the functor F. In particular, any representable map-functor is of the form A_K for some $K \subset F$. Moreover, for any map-functor A, the functor $\bar{A} := A_{K_A}$ is the smallest representable map-functor containing A.*

Proof. Obviously, $K \subset K_{A_K}$ and $A \subset A_{K_A} = \bar{A}$. Since the natural mapping $f: M \to F(M) \to F(M)/K(M) = N$ belongs to $A_K(M, N)$, we conclude that $K_{A_K} \subset K$. It remains to prove that $\bar{A}$ is contained in any representable map-functor B containing A. If B is represented by X, then the inclusion $B \subset \mathrm{Map}$ gives us an exact sequence of functors $0 \to K \hookrightarrow F \to X \to 0$. Consequently $B = A_K$ and hence $K \subset K_B \subset K_A$. Finally, $\bar{A} = A_{K_A} \subset A_K = B$. $\square$

An example for the last part of the lemma is the functor $\overline{\mathrm{Hom}}^m$ introduced in [7, Lemma 1.1 and Corollary 1.2]. This functor is, however, not satisfactory, because the representing functor $\bar{\Gamma}^m \subset \Gamma^m$, $\bar{\Gamma}^m(M) = R\{x^{(m)}: x \in M\}$, does not preserve Grothendieck sequences (see [7, Section 2]).

We will define 'satisfactory' map-functors as functors constituted by mappings f satisfying some equality conditions of the type

$$\sum_j r_j f\Big(\sum_k s_{jk} x_k\Big) = 0,$$

where r_j, s_{jk} ($j, k = 1, 2, \ldots$) denote fixed elements of R, almost all equal to zero, and x_k are arbitrary elements from the domain of f. For example homomorphisms, constants and quadratic mappings are of this kind.

More precisely, a class $\mathscr{A}$ of mappings between R-modules is called *equationally* *definable*, if $\mathscr{A}$ is the class of mappings $f: M \to N$ ($M, N \in R$-**Mod**) satisfying the following condition:

$$\forall_{x_k \in M} \sum_j r_{ij} f\Big(\sum_k s_{ijk} x_k\Big) = 0, \quad i \in I \tag{1.1}$$

for some fixed r_{ij}, $s_{ijk} \in R$ ($i \in I$, j, $k = 1, 2, \ldots$). Any equationally definable class $\mathscr{A}$ determines a map-functor A defined by $A(M, N) = \mathrm{Map}(M, N) \cap \mathscr{A}$ and called an *equationally definable functor* (shortly, an *ED-functor*).

Proposition 1.2. *For any map-functor A, the following conditions are equivalent*:

(1) *A is an* ED-*functor*;

(2) *A is represented by a functor preserving direct limits and Grothendieck sequences*;

(3) *$A = A_K$ where K preserves direct limits and epimorphisms.*

If the above are satisfied, then A is uniquely determined by $K(R^\infty)$, where $R^\infty = R \oplus R \oplus \cdots$.

Proof. (1) $\Rightarrow$ (2). Suppose that A is given by the class $\mathscr{A}$ satisfying (1.1). Then

$$K(M) = R\Big\{\sum_j r_{ij}\Big[\sum_k s_{ijk} x_k\Big] : i \in I, x_k \in M\Big\} \subset F(M)$$

defines a subfunctor K of F such that $A = A_K \simeq \mathrm{Hom}(F/K, -)$. We prove that F/K preserves (a) Grothendieck sequences and (b) direct limits.

(a) Let $M \underset{j}{\overset{i}{\rightrightarrows}} N \xrightarrow{q} P$ be a Grothendieck sequence. This means that i, j, q are R-homomorphisms, $q = \mathrm{Coker}(i, j)$ and

$$\forall_{x \in N} \exists_{t \in M}\, x = i(t) = j(t)$$

(see [7, p. 222]). The last condition is preserved by F/K since $\overline{[x]} = i_*(\overline{[t]}) = j_*(\overline{[t]})$. We must prove that $q_* = \mathrm{Coker}(i_*, j_*)$. It suffices to check that the sequence

$$0 \longrightarrow A(P, Q) \xrightarrow{A(q,1)} A(N, Q) \underset{A(j,1)}{\overset{A(i,1)}{\rightrightarrows}} A(M, Q)$$

is exact for any R-module Q. The only non-trivial part of this is the completion of the following commutative diagram:

$$\begin{tikzcd} M \arrow[r, shift left, "i"] \arrow[r, shift right, "j"'] \arrow[dr] & N \arrow[r, "q"] \arrow[d, "f"] & P \arrow[dl, dashed, "g"] \\ & Q & \end{tikzcd}$$

with $f \in \mathscr{A}$ by a mapping $g \in \mathscr{A}$. Observe that g exists and is defined by $g(q(n)) = f(n)$. Moreover, if $x_k \in P$, then $x_k = q(y_k)$ for some $y_k \in N$, and hence

$$\sum_j r_{ij} g\Big(\sum_k s_{ijk} x_k\Big) = \sum_j r_{ij}(gq)\Big(\sum_k s_{ijk} y_k\Big) = \sum_j r_{ij} f\Big(\sum_k s_{ijk} y_k\Big) = 0.$$

This proves that $g \in \mathscr{A}$.

(b) It suffices to prove that the natural homomorphism $\varrho : A(\varinjlim M_\alpha, N) \to \varprojlim A(M_\alpha, N)$ is an isomorphism. This is obvious for $A = \text{Map}$. It remains to prove that $f : M = \varinjlim M_\alpha \to N$ is in the class $\mathscr{A}$ provided that so are all mappings in $\varrho(f) = (f \circ \varrho_\alpha)_\alpha$. Since any sum in (1.1) is finite, we can assume that all $x_k \in M$ come from one M_α, that is, $x_k = \varrho_\alpha(y_k)$ for some $y_k \in M_\alpha$. In this case

$$\sum_j r_{ij} f\Big(\sum_k s_{ijk} x_k\Big) = \sum_j r_{ij}(f \circ \varrho_\alpha)\Big(\sum_k s_{ijk} y_k\Big) = 0,$$

as desired.

(2) $\Rightarrow$ (3). Since A is representable, it follows from Lemma 1.1 that $A = A_K \simeq \text{Hom}(F/K, —)$ for some subfunctor K of F. Moreover, F/K preserves direct limits and Grothendieck sequences. Since the functor $\varinjlim$ is exact and F preserves direct limits, it follows that so does K. It remains to prove that K preserves epimorphisms. For an epimorphism $q : N \to P$ we introduce a Grothendieck sequence $M \overset{i}{\underset{j}{\rightrightarrows}} N \xrightarrow{q} P$ with $M = N \oplus \ker(q)$, $i = (1, \hookrightarrow)$ and $j = (1, 0)$. Since F and F/K preserve Grothendieck sequences, we obtain the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
 & K(N) \arrow[r, "K(q)"] \arrow[d] & K(P) \arrow[d] & \\
F(M) \arrow[r, "F(i)-F(j)"] \arrow[d] & F(N) \arrow[r, "F(q)"] \arrow[d] & F(P) \arrow[r] \arrow[d] & 0 \\
F(M)/K(M) \arrow[r, "\overline{F(i)}-\overline{F(j)}"] \arrow[d] & F(N)/K(N) \arrow[r, "\overline{F(q)}"] \arrow[d] & F(P)/K(P) \arrow[r] \arrow[d] & 0 \\
0 & 0 & 0 &
\end{tikzcd}$$

An easy diagram consideration (in fact the Snake Lemma) shows that $K(q)$ is an epimorphism.

(3) ⇒ (1). Let

$$K(R^{\infty}) = R\left\{ \sum_j r_{ij}\left[\sum_k s_{ijk} e_k \right] : i \in I \right\}$$

for some r_{ij}, $s_{ijk} \in R$. For any R-module M and any $x_1, x_2, \ldots \in M$ consider an epimorphism $\varphi : R^{\infty} \to R\{x_1, x_2, \ldots\} \subset M$ given by $\varphi(e_k) = x_k$, $k = 1, 2, \ldots$ It follows from the assumption that

$$K(R\{x_1, x_2, \ldots\}) = K(\varphi)(K(R^{\infty})) = R\left\{ \sum_j r_{ij}\left[\sum_k s_{ijk} x_k \right] : i \in I \right\},$$

and consequently (since K is a subfunctor of F)

$$K(M) = \varinjlim_{\{x_1, x_2, \ldots\} \subset M} K(R\{x_1, x_2, \ldots\}) = R\left\{ \sum_j r_{ij}\left[\sum_k s_{ijk} x_k \right] : i \in I,\, x_k \in M \right\}.$$

Finally,

$$\begin{aligned} A(M, N) = A_K(M, N) &= \{ f \in \mathrm{Map}(M, N) : \bar{f}(K(M)) = 0 \} \\ &= \left\{ f \in \mathrm{Map}(M, N) : \forall_{i \in I}\, \forall_{x_k \in M} \sum_j r_{ij} f\left(\sum_k s_{ijk} x_k \right) = 0 \right\} \end{aligned}$$

for fixed r_{ij}, $s_{ijk} \in R$. □

Corollary 1.3. *Any map-functor isomorphic to an* ED-*functor is also an* ED-*functor.* □

The above fact allows us to generalize the concept of an ED-functor. This is, however, not used in the sequel.

2. Equationally definable cover

It follows directly from the definition that any intersection of ED-functors is also an ED-functor. Consequently,

Corollary 2.1. *Any map-functor A admits an equationally definable cover, that is, the smallest* ED-*functor* ED(A) *containing A. Obviously,* ED(A) = ED($\bar{A}$). □

Our problem is to find the functor ED(A) more explicitly. We can assume that A is representable, i.e., that $A = A_K$ for some $K \subset F$. First we need the following:

Proposition 2.2. *Let $\mathcal{F}$ denote the category of projective or of finitely generated free R-modules. Then any subfunctor K of $F|_{\mathcal{F}}$ can be uniquely extended to a subfunctor $\bar{K}$ of F preserving direct limits and epimorphisms. This gives us a natural one-to-one correspondence between all* ED-*functors and all subfunctors of $F|_{\mathcal{F}}$.*

Proof. Let $\mathscr{F}$ denote the category of finitely generated free R-modules, and let M be an R-module. Then M is a directed union of modules $\varphi_\alpha(P_\alpha)$ for some $P_\alpha \in \mathscr{F}$ and some $\varphi_\alpha : P_\alpha \to M$. Consequently, the value of $\bar{K}$ on M should be defined as

$$\bar{K}(M) = \bigcup_\alpha F(\varphi_\alpha)(K(P_\alpha)) \subset F(M).$$

We will show that this definition is correct.

First of all, let M be a directed union of $\psi_\beta(Q_\beta)$ for some $Q_\beta \in \mathscr{F}$ and some $\psi_\beta : Q_\beta \to M$. Since $\varphi_\alpha(P_\alpha)$ are finitely generated, for every α there exist some $\beta = \beta(\alpha)$ such that $\varphi_\alpha(P_\alpha) \subset \psi_\beta(Q_\beta)$. This gives us commutative diagrams

$$\begin{tikzcd} P_\alpha \arrow[rr] \arrow[dr, "\varphi_\alpha"'] & & Q_\beta \arrow[dl, "\psi_\beta"] \\ & M & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} K(P_\alpha) \arrow[rr] \arrow[d, phantom, "\cap"] & & K(Q_\beta) \arrow[d, phantom, "\cap"] \\ F(P_\alpha) \arrow[rr] \arrow[dr, "F(\varphi_\alpha)"'] & & F(Q_\beta) \arrow[dl, "F(\psi_\beta)"] \\ & F(M) & \end{tikzcd}$$

Consequently, $F(\varphi_\alpha)(K(P_\alpha)) \subset F(\psi_\beta)(K(Q_\beta))$. From this and the symmetric consideration we obtain that $\bigcup_\alpha F(\varphi_\alpha)(K(P_\alpha)) = \bigcup_\beta F(\psi_\beta)(K(Q_\beta))$.

Observe that $\bar{K}$ is a subfunctor of F by a similar consideration, because we can complete the following diagram:

$$\begin{tikzcd} P_\alpha \arrow[r, dashed] \arrow[d, "\varphi_\alpha"'] & Q_\beta \arrow[d, "\psi_\beta"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

for any α and some $\beta = \beta(\alpha)$. Moreover, $\bar{K}|_{\mathscr{F}} = K$.

Let $f : M \to N$ be an epimorphism and suppose that M is a directed union of $\varphi_\alpha(P_\alpha)$ for some $P_\alpha \in \mathscr{F}$. Then N is a directed union of modules $(f \circ \varphi_\alpha)(P_\alpha)$, and hence

$$\bar{K}(f)(\bar{K}(M)) = F(f)\Bigl(\bigcup_\alpha F(\varphi_\alpha)(K(P_\alpha))\Bigr) = \bigcup_\alpha F(f \circ \varphi_\alpha)(K(P_\alpha)) = \bar{K}(N).$$

Finally, let $M = \varinjlim M_\sigma$ and suppose that M_σ is a directed union of modules $\varphi_{\sigma,\alpha}(P_{\sigma,\alpha})$, $\alpha \in I(\sigma)$, where $P_{\sigma,\alpha} \in \mathscr{F}$. Then M is a directed union of $\psi_\sigma(M_\sigma)$, and hence

$$M = \bigcup_{(\sigma,\alpha)} (\psi_\sigma \varphi_{\sigma,\alpha})(P_{\sigma,\alpha}).$$

Since $P_{\sigma,\alpha}$ are finitely generated it follows that the above union is directed. Consequently,

$$\bar{K}(M) = \bigcup_{(\sigma,\alpha)} F(\psi_\sigma \varphi_{\sigma,\alpha})(K(P_{\sigma,\alpha})) = \bigcup_\sigma F(\psi_\sigma)\Bigl[\bigcup_{\alpha \in I(\sigma)} F(\varphi_{\sigma,\alpha})(K(P_{\sigma,\alpha}))\Bigr]$$

$$= \bigcup_\sigma \bar{K}(\psi_\sigma)(\bar{K}(M_\sigma)) = \varinjlim \bar{K}(M_\sigma),$$

since $\bar{K}$ is a subfunctor of F.

Let now $\mathscr{F}$ denote the category of projective R-modules and let P be a free R-module with a basis $\{e_t: t\in T\}$. For any finite subset $\sigma\subset T$ define $P_\sigma = R\{e_t: t\in\sigma\}$. Obviously, $P=\varinjlim P_\sigma$. Since P_σ is a retract of P it follows that $K(P_\sigma)=K(p_\sigma)K(P)$ where p_σ is the projection. If $K(P)=R\{\sum_j r_{ij}[\sum_{t\in T} s_{ijt}e_t]: i\in I\}$, then $K(P_\sigma)=R\{\sum_j r_{ij}[\sum_{t\in\sigma} s_{ijt}e_t]: i\in I\}$ and hence $K(P)=\varinjlim K(P_\sigma)$. The first part of the proof gives us an extension $\bar{K}$ of $K|_{\mathscr{F}'}$, where $\mathscr{F}'$ denotes the category of finitely generated free R-modules, such that $\bar{K}$ preserves direct limits and epimorphisms. Consequently, $\bar{K}(P)=K(P)$. Since any projective R-module is a retract of a free R-module it follows that $\bar{K}$ equals K on $\mathscr{F}$. $\square$

The above proposition allows us to prove the following characterizations of equationally definable covers:

Theorem 2.3. *If* $\mathrm{ED}(A_K)=A_{\bar{K}}$, *then* $\bar{K}=K$ *on the category of all projective R-modules. Consequently, if* $M=\varphi(P)$ *for some projective R-module P and some epimorphism* φ, *then*

(a) $\bar{K}(M)=F(\varphi)K(P)$,

(b) $A_{\bar{K}}(M,N)=\{f\in \mathrm{Map}(M,N): f\varphi\in A_K(P,N)\}\simeq\{g\in A_K(P,N): g(x+y)=g(x)$ *for* $x\in P,\ y\in\mathrm{Ker}(\varphi)\}$,

(c) $(F/\bar{K})(M)\simeq (F/K)(P)/R\{\overline{[x+y]}-\overline{[x]}: x\in P,\ y\in\mathrm{Ker}(\varphi)\}$.

Theorem 2.4. *If A is a map-functor and B is an* ED*-functor, then the following conditions are equivalent*:

(1) $B=\mathrm{ED}(A)$,

(2) K_B *and* K_A *coincide on the category of all projective R-modules*,

(3) $K_B(R^\infty)=K_A(R^\infty)$,

(4) $K_B(R^n)=K_A(R^n)$ *for any natural n.*

Moreover, if $K_A(R^\infty)=R\{\sum_j r_{ij}[\sum_k s_{ijk}e_k]: i\in I\}$, *then* $\mathrm{ED}(A)$ *is given by relations* $\sum_j r_{ij}f(\sum_k s_{ijk}x_k)=0,\ i\in I$.

Proof of Theorem 2.3. Let K be a subfunctor of F. It follows from Propostion 2.2 that there exists a functor $\bar{K}$ preserving direct limits and epimorphisms such that $\bar{K}=K$ on the category of projective R-modules. Let $M=\varphi(P)$ for some projective R-module P and some epimorphism φ. Then $\bar{K}(M)=F(\varphi)\bar{K}(P)=F(\varphi)K(P)\subset K(M)$, and hence $\bar{K}$ is a subfunctor of K. It is the greatest one between subfunctors L of K preserving direct limits and epimorphisms. In fact, for any such L, $L(M)=F(\varphi)L(P)\subset F(\varphi)K(P)=\bar{K}(M)$. In view of Lemma 1.1 and Proposition 1.2 the functor $A_{\bar{K}}$ is the smallest ED-functor containing A_K, that is, $A_{\bar{K}}=\mathrm{ED}(A_K)$.

To prove (b), we compute

$$\begin{aligned} A_{\bar{K}}(M,N) &= \{f\in\mathrm{Map}(M,N): \bar{f}(\bar{K}(M))=0\} \\ &= \{f\in\mathrm{Map}(M,N): \overline{(f\circ\varphi)}(K(P))=0\} \end{aligned}$$

$$= \{f \in \mathrm{Map}(M, N) \colon f \circ \varphi \in A_K(P, N)\}$$
$$\simeq \{g \in A_K(P, N) \colon g = f \circ \varphi \text{ for some } f \in \mathrm{Map}(M, N)\}$$
$$= \{g \in A_K(P, N) \colon g(x+y) = g(x) \text{ for } x \in P \text{ and } y \in \mathrm{Ker}(\varphi)\}.$$

Property (c) follows from (b) or from a suitable Grothendieck sequence (cf. Proposition 1.2(2)). □

Proof of Theorem 2.4. Observe that (1) ⇒ (2) by Theorem 2.3 since $B = A_{K_B}$ and $\mathrm{ED}(A) = \mathrm{ED}(A_{K_A})$, (2) ⇒ (3) is evident, and (3) ⇒ (4) since R^n is a retract of R^∞. We prove (4) ⇒ (1). It follows from the first implication that $K_B = K_A = K_{\mathrm{ED}(A)}$ on the category of finitely generated free R-modules. Since B and $\mathrm{ED}(A)$ are ED-functors, it follows from Proposition 2.2 that $B = \mathrm{ED}(A)$. The last part of the theorem follows from preceding equivalences and the proof of Proposition 1.2 ((3) ⇒ (1)). □

Let now $A = \mathrm{Hom}^m$. Since $\overline{\mathrm{Hom}}^m$ is represented by $F/K_A \simeq \bar{\Gamma}^m$ (see Section 1), Theorem 2.3(c) gives us the following:

Corollary 2.5. *If* $M = \varphi(P)$ *and* P *is projective, then* $(\mathrm{ED}(\mathrm{Hom}^m))(M, -)$ *is represented by the module*

$$\bar{\Gamma}^m(P)/R\{(x+y)^{(m)} - x^{(m)} \colon x \in P, y \in \mathrm{Ker}(\varphi)\}. \qquad \square$$

Corollary 2.6. *If* R *is a field, then* Hom_R^m *is an* ED*-functor.*

Proof. $\mathrm{Hom}_R^m = \overline{\mathrm{Hom}}_R^m$ since $\bar{\Gamma}_R^m(M)$ is a direct summand of $\Gamma_R^m(M)$ for any M (cf. [7, Section 1]), and $\overline{\mathrm{Hom}}_R^m = \mathrm{ED}(\overline{\mathrm{Hom}}_R^m)$ by Theorem 2.3. □

As another application of the above theorems, consider

Example 2.7. Let $A(M, N) = \{f \in \mathrm{Map}(M, N) \colon f(T(M)) = 0\}$, where $T(M)$ denotes the set of torsion elements in M. It is easy to see that $A = A_K$ where $K(M) = F(T(M))$. If R is a domain and M is projective, then, obviously, $K(M) = F(0) \simeq R$. Therefore $\mathrm{ED}(A) = A_{\bar{K}}$ where $\bar{K}(M) = F(0)$ and consequently

$$(\mathrm{ED}(A))(M, N) = \{f \in \mathrm{Map}(M, N) \colon f(0) = 0\}.$$

If R is arbitrary, then $K(R^\infty) = F(T(R^\infty)) = R\{[\sum_i r_i e_i] \colon rr_i = 0 \text{ for some } 0 \neq r \in R\}$ and hence $\mathrm{ED}(A)$ is given by the following conditions:

$$f\Big(\sum_i r_i x_i\Big) = 0 \quad \text{if } r_i \in \mathrm{Ann}(r) \text{ for some } 0 \neq r \in R.$$

For example, if any annihilator in R is principal, the relations are the following: $f(rx) = 0$ for all zero-divisors r of R.

3. n-covering relations

We introduce the defect decomposition of a functor following [2] or [7], and refer to those papers for details.

Let $K: R\text{-}\mathbf{Mod} \to R\text{-}\mathbf{Mod}$ be a functor. Consider the functors

$$K^n : \underbrace{R\text{-}\mathbf{Mod} \times \cdots \times R\text{-}\mathbf{Mod}}_{n} \to R\text{-}\mathbf{Mod}, \quad n = 0, 1, 2, \ldots,$$

defined by the formula

$$K^n(M_1, \ldots, M_n)$$
$$= \Big(\sum_{H \subset [1,n]} (-1)^{n-|H|} K(p_H) \Big)(K(M_1 \oplus \cdots \oplus M_n)) \subset K(M_1 \oplus \cdots \oplus M_n),$$

where $p_H \in \mathrm{End}(M_1 \oplus \cdots \oplus M_n)$ is the projection on $\bigoplus_{i \in H} M_i$. In particular, $K^0 = K(0)$ and $K^n(0, \ldots, 0) = 0$ for $n \geq 1$.

If K is a subfunctor of F, then we have exact sequences $0 \to K^n \hookrightarrow F^n \to (F/K)^n \to 0$ for all $n = 0, 1, 2, \ldots,$ and $K^n(M_1, \ldots, M_n) = K(M_1 \oplus \cdots \oplus M_n) \cap F^n(M_1, \ldots, M_n)$. Moreover, $F^n(M_1, \ldots, M_n)$ is generated by elements

$$(x_1, \ldots, x_n) := \sum_{H \subset [1,n]} (-1)^{n-|H|} \Big[\sum_{k \in H} x_k \Big] = (\Delta^n[\ ])(x_1, \ldots, x_n)$$

where $x_i \in M_i$, $i = 1, \ldots, n$. In particular, $(\ ) = [0]$ and $(x) = [x] - [0]$.

With the aid of functors introduced above we obtain the *defect decomposition*

$$K(M_1 \oplus \cdots \oplus M_n) = \bigoplus_{m=0}^{n} \bigoplus_{1 \leq j_1 < \cdots < j_m \leq n} K^m(M_{j_1}, \ldots, M_{j_m}). \tag{3.1}$$

In particular, $K(R^n)$ is uniquely determined by $K^m(R) := K^m(R, \ldots, R)$ for $m = 0, \ldots, n$. Consequently, Theorem 2.4 gives us the following:

Corollary 3.1. *If A is a map-functor and B is an* ED*-functor, then $B = \mathrm{ED}(A)$ iff $K_B^n(R) = K_A^n(R)$ for any natural n.* □

We say that an ED-functor B containing A is an *n-covering functor of A* if $K_B^n(R) = K_A^n(R)$. In this case, relations defining B are called *n-covering relations of A*. It follows from the above corollary that an ED-functor B is an n-covering functor of A for any natural n iff $B = \mathrm{ED}(A)$.

The last sentence of Theorem 2.4 admits the following generalization allowing us to obtain n-covering relations of a map-functor A:

Proposition 3.2. *Let $B = \mathrm{ED}(A)$ and $K = K_A$.*

(a) *If B is given by condition* (1.1), *then*

$$K(R^{\infty})=R\left\{\sum_j r_{ij}\left[\sum_m\left(\sum_k s_{ijk}t_{km})e_m\right]: i\in I,\ t_{km}\in R\right\}.$$

(b) *If* $K(R^{\infty})=R\{\sum_j r_{ij}[\sum_k s_{ijk}e_k]:\ i\in I\}$, *then for any natural* n

$$K^n(R)=R\left\{\sum_j r_{ij}(s_{ij1}e_1,\dots,s_{ijn}e_n):\ i\in I\right\}.$$

(c) *If* $K^n(R)=R\{\sum_j r_{ij}^{(n)}(s_{ij1}^{(n)}e_1,\dots,s_{ijn}^{(n)}e_n):\ i\in I_n\}$, *then*

$$\sum_j r_{ij}^{(n)}(\Delta^n f)(s_{ij1}^{(n)}x_1,\dots,s_{ijn}^{(n)}x_n)=0,\ i\in I_n \tag{B_n}$$

is a system of n-covering relations of A and B. In particular, B is given by relations (B_n), $n=0,1,\dots$.

Proof. (a)

$$K(R^{\infty})=K_B(R^{\infty})=R\left\{\sum_j r_{ij}\left[\sum_k s_{ijk}x_k\right]: i\in I,\ x_k\in R^{\infty}\right\}$$
$$=R\left\{\sum_j r_{ij}\left[\sum_{k,m} s_{ijk}t_{km}e_m\right]: i\in I,\ t_{km}\in R\right\}.$$

(b) Let $n\geq 0$. Observe that $K(R^n)=R\{\sum_j r_{ij}[\sum_{k=1}^n s_{ijk}e_k]: i\in I\}$ and that $\sum_j r_{ij}[\sum_{k\in H}s_{ijk}e_k]\in K(R^n)$ for any $i\in I$ and $H\subset[1,n]$. Hence $\sum_j r_{ij}(s_{ij1}e_1,\dots,s_{ijn}e_n)\in K(R^n)\cap F^n(R)=K^n(R)$ for $i\in I$. In the same way, $\sum_j r_{ij}(s_{ijj_1}e_{j_1},\dots,s_{ijj_m}e_{j_m})\in K^m(Re_{j_1},\dots,Re_{j_m})$ for $1\leq j_1<\dots<j_m\leq n$. On the other hand

$$[x_1+\dots+x_n]=\sum_{m=0}^n\ \sum_{1\leq j_1<\dots<j_m\leq n}(x_{j_1},\dots,x_{j_m}),$$

and hence

$$\sum_j r_{ij}\left[\sum_{k=1}^n s_{ijk}e_k\right]=\sum_{m=0}^n\ \sum_{1\leq j_1<\dots<j_m\leq n}\left(\sum_j r_{ij}(s_{ijj_1}e_{j_1},\dots,s_{ijj_m}e_{j_m})\right).$$

It suffices to observe that this is the presentation of a generator in the direct decomposition (3.1) where $M_i=Re_i$.

(c) Let C be an ED-functor given by relations (B_n) (for a fixed n). Since

$$K_C(R^n)=R\left\{\sum_j r_{ij}^{(n)}(s_{ij1}^{(n)}x_1,\dots,s_{ijn}^{(n)}x_n);\ i\in I_n,\ x\in R^n\right\},$$

it follows that $K^n(R)\subset K_C(R^n)\cap F^n(R)=K_C^n(R)$. On the other hand, $K^n(R)\subset K(R^n)\subset K(R^{\infty})$, and hence mappings from B satisfy relations (B_n) by Theorem 2.4. Consequently, $A\subset B\subset C$ and therefore $K_C^n(R)\subset K_B^n(R)\subset K^n(R)$. This proves that $K^n(R)=K_B^n(R)=K_C^n(R)$. □

Observe that the complete system of relations (B_n), $n=0,1,\dots$ obtained above for an ED-functor B is 'too large' in general, because some (B_n) can be m-covering for another m. This follows from the following:

Example 3.3. Let $m \geq 0$ and

$$B(M,N)=\{f\in \mathrm{Map}(M,N): f(rx)=r^m f(x) \text{ for } r\in R,\, x\in M\}.$$

Then

$$K_B(R^\infty)=R\{[rx]-r^m[x]:\ r\in R,\, x\in R^\infty\}$$

$$=R\left\{\left[\sum_m (rt_m)e_m\right]-r^m\left[\sum_m t_m e_m\right]:\ r\in R,\, t_m\in R\right\},$$

$$K_B^n(R)=R\{(rt_1e_1,\dots,rt_ne_n)-r^m(t_1e_1,\dots,t_ne_n):\ r\in R,\, t_m\in R\},$$

and hence n-covering relations of B are the following:

(B_n) $$(\Delta^n f)(rt_1x_1,\dots,rt_nx_n)=r^m(\Delta^n f)(t_1x_1,\dots,t_nx_n),\quad r,\, t_m\in R,$$

or, equivalently:

(B_n') $$(\Delta^n f)(rx_1,\dots,rx_n)=r^m(\Delta^n f)(x_1,\dots,x_n),\quad r\in R.$$

However, the system of relations

(B_0') $$f(0)=r^m f(0),\quad r\in R,$$

(B_1') $$f(rx)-f(0)=r^m(f(x)-f(0)),\quad r\in R$$

is equivalent to the defining system, and hence is n-covering for any natural n.

We will find some n-covering functors of Hom^m. First we need the following general lemma:

Lemma 3.4. *Let B be an* ED*-functor containing A. Then B is an n-covering functor of A iff the natural epimorphism $\nu^n(R):(F/K_B)^n(R)\to(F/K_A)^n(R)$ is an isomorphism.*

Proof. This follows from the Snake Lemma applied to the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_B^n(R) & \longrightarrow & F^n(R) & \longrightarrow & (F/K_B)^n(R) & \longrightarrow & 0\\
 & & \big\downarrow\cap & & \big\| & & \big\downarrow \nu^n(R) & & \\
0 & \longrightarrow & K_A^n(R) & \longrightarrow & F^n(R) & \longrightarrow & (F/K_A)^n(R) & \longrightarrow & 0
\end{array}$$

□

Let $A = \mathrm{Hom}^m$ and $B = \mathrm{Appl}^m$ (see the introduction). The relations (A1), (A2) defining B can be rewritten as the following conditions:

$$(\mathrm{E}_1) \qquad f(rx) = r^m f(x), \quad r \in R,$$

$$(\mathrm{E}_m) \qquad (\Delta^m f)(rx, —) = r(\Delta^m f)(x, —), \quad r \in R,$$

$$(\mathrm{E}_{m+1}) \qquad (\Delta^{m+1} f)(x_1, \ldots, x_{m+1}) = 0.$$

It is known that B contains A and $\overline{\mathrm{Hom}}^m$ is represented by $\bar{\Gamma}^m \subset \Gamma^m$, $\bar{\Gamma}^m(M) = R\{x^{(m)}: x \in M\}$ (see [7]). Hence $F/K_A \simeq \bar{\Gamma}^m$. Let us define $\Delta^m(M) = F(M)/K_B(M) = R\{\delta^m(x): x \in M\}$, where $\delta^m(x) = \overline{[x]}$ (see [7]). There exists a natural epimorphism $h^m(M) : \Delta^m(M) \to \bar{\Gamma}^m(M)$, $\delta^m(x) \mapsto x^{(m)}$, and the homomorphism $v^n(R)$ from Lemma 3.4 can be rewritten as $h^{m,n}(R) : \Delta^{m,n}(R) \to \bar{\Gamma}^{m,n}(R)$ (see [7]). Consequently, Appl^m is an n-covering functor of Hom^m iff $h^{m,n}(R)$ is an isomorphism. The last condition was investigated in [7] and [8]. Corollaries 2.6 and 4.2 of [7] give us the following:

Theorem 3.5. Appl^m *is an n-covering functor* (*and hence* (E_1), (E_m), (E_{m+1}) *above are n-covering relations*) *of* Hom^m *for* $n = 0$, 1 *and* $n \geq m$. □

It is proved in [8] that $\mathrm{Ker}(h^{m,m-1}(R))$ is generated by elements

$$(\Delta^{m-1}\delta^m)(re_1, se_2, e_3, \ldots, e_{m-1}) - r(\Delta^{m-1}\delta^m)(e_1, se_2, e_3, \ldots, e_{m-1})$$
$$- s(\Delta^{m-1}\delta^m)(re_1, e_2, \ldots, e_{m-1}) + rs(\Delta^{m-1}\delta^m)(e_1, \ldots, e_{m-1}), \quad r, s \in R.$$

The corresponding relation is the following:

$$(\mathrm{E}_{m-1}) \qquad (\Delta^{m-1} f)(rx, sy, —) - r(\Delta^{m-1} f)(x, sy, —)$$
$$- s(\Delta^{m-1} f)(rx, y, —) + rs(\Delta^{m-1} f)(x, y, —)$$
$$= 0, \quad r, s \in R.$$

Let us define the functor of *regular m-applications* $C = \mathbf{Appl}^m \subset \mathrm{Appl}^m$ as an ED-functor given by conditions (E_1), (E_{m-1}), (E_m), (E_{m+1}). It follows from the above that $v^n(R) : (F/K_C)^n(R) \to (F/K_A)^n(R)$ is an isomorphism, and hence Lemma 3.4 gives us the following:

Theorem 3.6. $\mathbf{Appl}^m$ *is an n-covering functor* (*and hence* (E_1), (E_{m-1}), (E_m), (E_{m+1}) *above are n-covering relations*) *of* Hom^m *for* $n = 0$, 1 *and* $n \geq m - 1$. *In particular,* $\mathrm{ED}(\mathrm{Hom}^m) = \mathbf{Appl}^m$ *for* $m \leq 3$. □

As follows from the above, the description of n-covering relations of Hom^m is closely related to the description of $\mathrm{Ker}(h^{m,n}(R))$. The problem is still open for $1 < n < m - 1$, however, it is possible to give an information about the type of n-covering relations for those n. This is done in the next sections.

4. Strong ED-functors

Let S be a (commutative) ring and let S-**Alg** denote the category of all (commutative) S-algebras. A subfunctor A of Map defined on the disjoint union of the categories $R\text{-}\mathbf{Mod}^0 \times R\text{-}\mathbf{Mod}$ ($R \in S\text{-}\mathbf{Alg}$) will be called a *map-functor over S-algebras* if K_A is a subfunctor of F on the (full) category $M(S)$ of pairs (R, M) where $R \in S\text{-}\mathbf{Alg}$ and $M \in R\text{-}\mathbf{Mod}$. In this case, we can define the functor $K_A^n : S\text{-}\mathbf{Alg} \to M(S)$, $K_A^n(R) = (R, K_A^n(R, \ldots, R))$, which is, obviously, a sub-functor of F^n. A map-functor over S-algebras which is an ED-functor over any $R \in S\text{-}\mathbf{Alg}$ will be called an ED-*functor over S-algebras.*

Lemma 4.1. *If A is a map-functor over S-algebras, then so is* ED(A).

Proof. Let us denote $K = K_A$ and $\bar{K} = K_{\mathrm{ED}(A)}$. If $(\varphi, \psi) : (R, M) \to (R', M')$ in $M(S)$, then there exists a commutative diagram

$$\begin{tikzcd}
(R, P) \arrow[r, "(\varphi, \eta)"] \arrow[d, "(1, \alpha)"'] & (R', P') \arrow[d, "(1, \beta)"] \\
(R, M) \arrow[r, "(\varphi, \psi)"] & (R', M')
\end{tikzcd}$$

where P and P' are projective over R and R', respectively, and α, β are epimorphisms. Hence

$$\begin{aligned}
F(\varphi, w)(\bar{K}(R, M)) &= F(\varphi, \psi)F(1, \alpha)(K(R, P)) \\
&= F(1, \beta)F(\varphi, \eta)(K(R, P)) \subset F(1, \beta)(K(R', P')) \\
&= \bar{K}(R', M')
\end{aligned}$$

by Theorem 2.3. $\square$

A class $\mathscr{A}$ of mappings $f : M \to N$ ($M, N \in R\text{-}\mathbf{Mod}$, $R \in S\text{-}\mathbf{Alg}$) is called *strongly equationally definable* (over S) if $\mathscr{A}$ is given by the following conditions:

$$\forall_{a_k \in R} \forall_{x_k \in M} \sum_j r_{ij}(a_1, a_2, \ldots) f\Big(\sum_k s_{ijk}(a_1, a_2, \ldots) x_k\Big) = 0, \quad i \in I \tag{4.1}$$

for some fixed r_{ij}, $s_{ijk} \in S[T_1, T_2, \ldots]$ ($i \in I$, j, $k = 1, 2, \ldots$), almost all equal to zero for each fixed $i \in I$. Relations of the type (4.1) are called *strong* (over S). Any strongly equationally definable class $\mathscr{A}$ determines a functor A defined by $A(M, N) = \mathrm{Map}(M, N) \cap \mathscr{A}$ and called a *strong* ED-*functor* (over S).

Lemma 4.2. *Any strong* ED-*functor over S is an* ED-*functor over S-algebras.*

Proof. It suffices to prove that $K=K_A$ is a subfunctor of F on the category $M(S)$ provided that A is a functor given by conditions (4.1). Obviously,

$$K(R,M)=R\left\{\sum_j r_{ij}(a_1,a_2,\dots)\left[\sum_k s_{ijk}(a_1,a_2,\dots)x_k\right]: i\in I,\ a_k\in R,\ x_k\in M\right\}.$$

Recall that F is a functor on $M(S)$ in the following way: if $(\varphi,\psi):(R,M)\to(R',M')$, then $F(\varphi,\psi):F(R,M)\to F(R',M')$, $F(\varphi,\psi)(\sum_j r_j[x_j])=\sum_j \varphi(r_j)[\psi(x_j)]$. Then

$$\begin{aligned}&F(\varphi,\psi)\left(\sum_j r_{ij}(a_1,a_2,\dots)\left[\sum_k s_{ijk}(a_1,a_2,\dots)x_k\right]\right)\\ &\quad=\sum_j r_{ij}(\varphi(a_1),\varphi(a_2),\dots)\left[\sum_k s_{ijk}(\varphi(a_1),\varphi(a_2),\dots)\psi(x_k)\right]\in K(R',M'),\end{aligned}\tag{4.2}$$

as desired. □

Proposition 4.3. *Let $A=A_K$ be an* ED*-functor over S-algebras. Then the following conditions are equivalent*:

(1) *A is a strong* ED*-functor over S*;

(2) *$K(R,M)\to K(R/I,N)$ is an epimorphism for any S-algebra R, any ideal $I\subset R$ and any epimorphism $M\to N$ over $R\to R/I$*;

(3) *$K(R,R^n)\to K(R/I,(R/I)^n)$ is an epimorphism for any S-algebra R, any ideal $I\subset R$ and any natural n*;

(4) *$K(R,R^\infty)\to K(R/I,(R/I)^\infty)$ is an epimorphism for any S-algebra R and any ideal $I\subset R$.*

Proof. Observe that (1) ⇒ (2) by formula (4.2) and (2) ⇒ (3) ⇒ (4) since $K(R,—)$ and $K(R/I,—)$ commute with direct limits. We prove (4) ⇒ (1). Let $R'=S[T_1,T_2,\dots]$ and suppose that

$$K(R',R'^{\infty})=R'\left\{\sum_j r_{ij}(T_1,T_2,\dots)\left[\sum_k s_{ijk}(T_1,T_2,\dots)e'_k\right]: i\in I\right\}.$$

Then for any S-algebra R we have

$$K(R,R^\infty)\supset R\left\{\sum_j r_{ij}(a_1,a_2,\dots)\left[\sum_k s_{ijk}(a_1,a_2,\dots)e_k\right]: i\in I,\ a_k\in R\right\}.$$

In virtue of Theorem 2.4 it suffices to prove that the above inclusion is in fact an equality. Observe that $R\simeq R''/I$ where $R''=S[T_\sigma]_{\sigma\in\Sigma}$ and Σ is infinite. By (4), any element $x\in K(R,R^\infty)$ is an image of an element $y\in K(R'',R''^{\infty})$, which is, obviously, of the form

$$y=\sum_j v_j(T_{\sigma_1},T_{\sigma_2},\dots)\left[\sum_k w_{jk}(T_{\sigma_1},T_{\sigma_2},\dots)e''_k\right].$$

Since $R' \simeq S[T_{\sigma_1}, T_{\sigma_2}, \ldots]$ is a retract of R'', it follows that x belongs to the image of the composition $K(R', R'^{\infty}) \hookrightarrow K(R'', R''^{\infty}) \to K(R, R^{\infty})$ induced by $R' \hookrightarrow R'' \to R$, and hence x belongs to the above submodule of $K(R, R^{\infty})$. □

Example 4.4. The functor B given by relations

$$f(x) = rf(x) \quad \text{if } r^2 = -1$$

is an ED-functor over all rings which is not a strong ED-functor by Proposition 4.3. In fact, $K_B(\mathbb{R}[T], 0) = 0$ and $K_B(\mathbb{C}, 0) = \mathbb{C}\{[0] \pm \mathrm{i}[0]\} \neq 0$. Moreover, B is not a strong ED-functor over S for any ring S. The functors A and $\mathrm{ED}(A)$ from Example 2.7 are not even map-functors over S-algebras (for any S).

Let A be a map-functor over S-algebras. We say that A is *n-strong* (over S) if for any S-algebra R the module $K_A^n(R)$ is generated by elements of the form

$$\sum_j r_{ij}(a_1, a_2, \ldots)(s_{ij1}(a_1, a_2, \ldots)e_1, \ldots, s_{ijn}(a_1, a_2, \ldots)e_n) \quad (i \in I)$$

for some fixed r_{ij}, $s_{ijk} \in S[T_1, T_2, \ldots]$. A is called *strong* (over S) if A is n-strong (over S) for any natural n. Proposition 3.2 shows the conformity of the double meaning of the expression 'strong ED-functor'.

Proposition 4.5. *If A is a map-functor over S-algebras, then the following conditions are equivalent*:

(1) *A is n-strong*;

(2) $\mathrm{ED}(A)$ *is n-strong*;

(3) *$K_A^n(R) \to K_A^n(R/I)$ is an epimorphism for any S-algebra R and any ideal $I \subset R$*;

(4) *A admits a system of strong n-covering relations.*

Proof. (1) is equivalent to (2) since $K_A^n(R) = K_{\mathrm{ED}(A)}^n(R)$ for any S-algebra R, and this is equivalent to (3) as in the proof of Proposition 4.3 (we consider $K_A^n(R)$ instead of $K(R, R^{\infty})$). Condition (4) means that there exists a strong ED-functor B containing A such that $K_B^n(R) = K_A^n(R)$ for any S-algebra R, and this is equivalent to (2) by Proposition 3.2 (we treat $\{t_{km}\}$ as an independent countable set of variables over S). □

Let us consider functors related to polynomial mappings. The above consideration gives us the following:

Proposition 4.6. (1) Appl^m *and* $\mathbf{Appl}^m$ *are strong* ED-*functors over* $\mathbb{Z}$ *for any natural m.*

(2) Hom^m, $\overline{\mathrm{Hom}}^m$ *and* $\mathrm{ED}(\mathrm{Hom}^m)$ *are map-functors over $\mathbb{Z}$-algebras which are n-strong for $n = 0$, 1 and $n \geq m-1$.*

Proof. Part (1) is evident. To prove (2), denote $\overline{\mathrm{Hom}}^m = A_K$. Then $F/K \simeq \bar{\Gamma}^m$ where $\bar{\Gamma}^m$ is a functor on $M(\mathbb{Z})$. In fact, if $(\varphi, \psi):(R, M) \to (R', M')$, then $\bar{\Gamma}^m(\varphi, \psi)$: $\bar{\Gamma}_R^m(M) \to \bar{\Gamma}_{R'}^m(M')$, $x^{(m)} \mapsto (\psi(x))^{(m)}$. Hence K is a subfunctor of F on $M(\mathbb{Z})$ and consequently Hom^m and $\overline{\mathrm{Hom}}^m$ are map-functors over $\mathbb{Z}$-algebras. By Lemma 4.1, so is also $\mathrm{ED}(\mathrm{Hom}^m)$. The rest follows from (1) and Theorem 3.6. □

In Section 6 we determine all numbers m, n for which Hom^m (equivalently, $\overline{\mathrm{Hom}}^m$ or $\mathrm{ED}(\mathrm{Hom}^m)$) is n-strong. They are the following:

(a) For all rings (i.e., for $S = \mathbb{Z}$): $n \le 1$ or $m \le 5$ or $m - n \le 1$,

(b) For rings with $\frac{1}{2}$ (i.e., for $S = \mathbb{Z}[\frac{1}{2}]$): $n \le 1$ or $m \le 7$ or $m - n \le 3$.

In the case of $S = \mathbb{Z}$ we know strong n-covering relations (E_n) of Hom^m for $n \le 1$ and $m - n \le 1$ (Theorem 3.6). The remaining strong relations:

$$(\mathrm{E}_2) \text{ for } m = 4, 5 \quad \text{and} \quad (\mathrm{E}_3) \text{ for } m = 5$$

are still unknown.

5. The structure of $\tilde{\Gamma}^m$

This section gives auxiliary results needed for the proof of the Main Theorem in Section 6. It is, however, of independent interest from the point of view of [5], [6] and [7].

Let us consider the functors $\tilde{\Gamma}^m = \Gamma^m / \bar{\Gamma}^m$ and their defects $\tilde{\Gamma}^{m,n} = \Gamma^{m,n} / \bar{\Gamma}^{m,n}$. We can assume that $m, n > 0$ since otherwise $\tilde{\Gamma}^{m,n} = 0$. It follows from [6] or [7] that

$$\tilde{\Gamma}^{m,n}(R) \simeq \bigoplus_{M \in \mathrm{Max}(R)} \tilde{\Gamma}^{m,n}(R/M^{k(M)})$$

for a noetherian ring R and all sufficiently large $k(M)$. Since $\tilde{\Gamma}^m$ commutes with localizations, we can restrict our consideration to local noetherian rings (R, M). In this case we obtain $\tilde{\Gamma}^{m,n}(R) \simeq \tilde{\Gamma}^{m,n}(R/M^{k(M)})$ or, equivalently, $M^{k(M)} \tilde{\Gamma}^{m,n}(R) = 0$. It is known (see [6, Corollary 3.3]) that we can take $k(M) = 1$ if $m \le 5$ (or $m \le 7$ and 2 is invertible in R). The following theorem explains this situation in a more general (in particular, not necessarily noetherian) context:

Theorem 5.1. *If K is a field with q elements, then the following conditions are equivalent*:

(1) $M\tilde{\Gamma}^{m,n}(R) = 0$ *for each local ring* (R, M) *such that* $R/M \simeq K$;

(2) $M\tilde{\Gamma}^{m,n}(R) = 0$ *for some local ring* (R, M) *such that* $R/M \simeq K$ *and* $M \ne M^2$;

(3) $n = 1$ *or* $m < 2(q+1)$ *or* $m - n < 2(q-1)$;

(4) *if* $f \in K[T_1, \dots, T_n]$ *is a form of degree* m *divisible by* $T_1 \cdots T_n$ *and* f, $\partial f / \partial T_1, \dots, \partial f / \partial T_n$ *vanish as mappings, then* $f = 0$;

(5) $T\tilde{\Gamma}^{m,n}(K[T]/(T^2)) = 0$;

(6) $T\tilde{\Gamma}^{m,n}(R[T]/(T^2)) = 0$ *for each local ring* (R, M) *such that* $R/M \simeq K$.

For the proof, we need the following:

Lemma 5.2. *If $|K|=q$, then any polynomial $f\in K[T_1,\dots,T_n]$ vanishing on $(K^*)^n$ is of the form $f=\sum_{i=1}^n (T_i^{q-1}-1)f_i$ where $\deg(f_i)\le\deg(f)-(q-1)$. If, moreover, f is a form of degree m, then $f=\sum_{i=1}^{n-1}(T_i^{q-1}-T_n^{q-1})g_i$ where g_i are forms of degree $m-(q-1)$.*

Proof. Observe that

$$T_1^{i_1}\cdots T_n^{i_n}=T_1^{i_1-(q-1)}T_2^{i_2}\cdots T_n^{i_n}+(T_1^{q-1}-1)\,T_1^{i_1-(q-1)}T_2^{i_2}\cdots T_n^{i_n}=\cdots$$
$$=T_1^{j_1}\cdots T_n^{j_n}+\sum_{i=1}^{n}(T_i^{q-1}-1)f_i'$$

where $0\le j_1,\dots,j_n<q-1$ and $\deg(f_i')\le(i_1+\cdots+i_n)-(q-1)$. Consequently,

$$f=\sum_{i=1}^{n}(T_i^{q-1}-1)f_i+h,\quad \deg(f_i)\le\deg(f)-(q-1),\ \deg_{T_i}(h)<q-1,\ i=1,\dots,n.$$

By assumption, h vanishes on $(K^*)^n$. An easy induction on n shows that any such polynomial h with $\deg_{T_i}(h)<q-1$ for $i=1,\dots,n$ must be zero. Now, let f be a form of degree m. Then $f=T_n^m g(T_1/T_n,\dots,T_{n-1}/T_n)$ where $g=f(T_1,\dots,T_{n-1},1)\in K[T_1,\dots,T_{n-1}]$. The first part of the lemma shows that $g=\sum_{i=1}^{n-1}(T_i^{q-1}-1)f_i$ where $\deg(f_i)\le m-(q-1)$. Consequently,

$$f=\sum_{i=1}^{n-1}T_n^m\left(\frac{T_i^{q-1}}{T_n^{q-1}}-1\right)f_i\left(\frac{T_1}{T_n},\dots,\frac{T_{n-1}}{T_n}\right)=\sum_{i=1}^{n-1}(T_i^{q-1}-T_n^{q-1})g_i$$

where $g=T_n^{m-(q-1)}f_i(T_1/T_n,\dots,T_{n-1}/T_n)$ are forms of degree $m-(q-1)$. □

Proof of Theorem 5.1. (1) ⇒ (2) is evident.

(2) ⇒ (3). Suppose that $n\ge2$, $m\ge2(q+1)$ and $m-n\ge2(q-1)$. Let us define the form $f\in R[T_1,\dots,T_n]$ of degree m in the following way:

$$f=\begin{cases}T_1^{m-n-2q}(T_1^qT_2-T_1T_2^q)^2T_3\cdots T_n & \text{if } m-n\ge2q,\\ (T_1^qT_2-T_1T_2^q)(T_2^qT_3-T_2T_3^q)T_4\cdots T_n & \text{if } m-n\ge2q-1 \text{ (and hence } n\ge3),\\ (T_1^qT_2-T_1T_2^q)(T_3^qT_4-T_3T_4^q)T_5\cdots T_n & \text{if } m-n\ge2q-2 \text{ (and hence } n\ge4).\end{cases}$$

Observe that f induces a homomorphism $g:\Gamma_R^m(R^n)\to R$ defined by $g(e_1^{(i_1)}\cdots e_n^{(i_n)})=$ the coefficient of f at $T_1^{i_1}\cdots T_n^{i_n}$ (see [9]). Since $g((r_1e_1+\cdots+r_ne_n)^{(m)})=f(r_1,\dots,r_n)$ and $r^qs-rs^q\in M$ for $r,s\in R$, it follows that $g(\bar\Gamma_R^m(R^n))\subset M^2$. Moreover, since $\Gamma^{m,n}(R)$ is generated by $e_1^{(i_1)}\cdots e_n^{(i_n)}$ for $i_1+\cdots+i_n=m$ and $i_1,\dots,i_n>0$, we obtain that $g(\Gamma^{m,n}(R))=R$. Consequently, g induces an epimorphism $h:\tilde\Gamma^{m,n}(R)=\Gamma^{m,n}(R)/\Gamma^{m,n}(R)\cap\bar\Gamma^m(R^n)\to R/M^2$. Since $M\tilde\Gamma^{m,n}(R)=0$ it follows that $M/M^2=0$, contradiction.

(3) ⇒ (4). Let $f=T_1\cdots T_ng$ where g is a form of degree $m-n$. Then

$$\frac{\partial f}{\partial T_j} = T_1 \cdots \hat{T}_j \cdots T_n g + T_1 \cdots T_n \frac{\partial g}{\partial T_j}, \quad j = 1, \dots, n.$$

It follows from the assumption that g and $\partial g/\partial T_j$ vanish on $(K^*)^n$, and, moreover, that $g(r_1, \dots, r_n) = 0$ if at most one r_i is zero. By Lemma 5.2,

$$g = \sum_{i=1}^{n-1} (T_i^{q-1} - T_n^{q-1}) g_i$$

where g_i are forms of degree $(m-n)-(q-1)$. Consequently, for $j<n$,

$$\frac{\partial g}{\partial T_j} = \sum_{i=1}^{n-1} (T_i^{q-1} - T_n^{q-1}) \frac{\partial g_i}{\partial T_j} - T_j^{q-2} g_j,$$

and hence $g_1, \dots, g_{n-1}$ also vanish on $(K^*)^n$. Therefore, by Lemma 5.2,

$$g = \sum_{i,j=1}^{n-1} (T_i^{q-1} - T_n^{q-1})(T_j^{q-1} - T_n^{q-1}) g_{ij}$$

where g_{ij} are forms of degree $(m-n)-2(q-1)$. Consequently, the condition $m-n<2(q-1)$ gives us $g=0$. Moreover, if $n=1$, then $f=aT_1^m$ and hence $a=f(1)=0$. It remains to assume that $m<2(q+1)$, $n\geq 2$ and $m-n\geq 2(q-1)$. Then $2q\leq m\leq 2q+1$ and we must consider the following three cases:

(a) $m=2q$, $n=2$. Then $g=a(T_1^{q-1}-T_2^{q-1})^2$ and $a=g(1,0)=0$.

(b) $m=2q+1$, $n=2$. Then $g=(T_1^{q-1}-T_2^{q-1})^2(aT_1+bT_2)$ where $a=g(1,0)=0$ and $b=g(0,1)=0$, consequently, $g=0$.

(c) $m=2q+1$, $n=3$. Then $g=a(T_1^{q-1}-T_3^{q-1})^2+b(T_1^{q-1}-T_3^{q-1})(T_2^{q-1}-T_3^{q-1})+c(T_2^{q-1}-T_3^{q-1})^2$ where $a=g(0,1,1)=0$, $c=g(1,0,1)=0$ and $a+b+c=g(1,1,0)=0$. Consequently, $g=0$.

(4) $\Rightarrow$ (5). Let $R=K[T]/(T^2)=K\oplus Kt$ where $t^2=0$. Then $\bar{\Gamma}^{m,n}(R)$ is generated by elements

$$\sum_{(i)} (s_1+s_1't)^{i_1} \cdots (s_n+s_n't)^{i_n} e_1^{(i_1)} \cdots e_n^{(i_n)}$$
$$= \sum_{(i)} s_1^{i_1} \cdots s_n^{i_n} e_1^{(i_1)} \cdots e_n^{(i_n)} + t \sum_{k=1}^{n} \sum_{(i)} i_k s_1^{i_1} \cdots s_k^{i_k-1} s_k' \cdots s_n^{i_n} e_1^{(i_1)} \cdots e_n^{(i_n)}$$

for $s_j, s_j' \in K$, where (i) passes through systems $(i_1, \dots, i_n)$ satisfying $i_1 + \cdots + i_n = m$ and $i_1, \dots, i_n > 0$. The first summand is a generator of $\bar{\Gamma}^{m,n}(K) \subset \bar{\Gamma}^{m,n}(R)$, and the sum

$$\sum_{(i)} i_k s_1^{i_1} \cdots s_k^{i_k-1} \cdots s_n^{i_n} e_1^{(i_1)} \cdots e_n^{(i_n)}$$

is also a generator of $\bar{\Gamma}^{m,n}(R)$ for any $k=1, \dots, n$ (put $s_j'=\delta_{jk}$). We must prove that $t\Gamma^{m,n}(R) \subset \bar{\Gamma}^{m,n}(R)$, and, for this goal, it suffices to check that

$$e_1^{(j_1)} \cdots e_n^{(j_n)} \in N$$

$$:= \bar{\Gamma}^{m,n}(K) + K\left\{ \sum_{(i)} i_k s_1^{i_1} \cdots s_k^{i_k-1} \cdots s_n^{i_n} e_1^{(i_1)} \cdots e_n^{(i_n)}, \quad k = 1, \dots, n \right\}.$$

In other words, we will prove that $\Gamma^{m,n}(K) = N$, or, equivalently, that any K-homomorphism $g : \Gamma^{m,n}(K) \to K$ vanishing on N is zero. Consider the polynomial

$$f = \sum_{(i)} g(e_1^{(i_1)} \cdots e_n^{(i_n)})\, T_1^{i_1} \cdots T_n^{i_n} \in K[T_1, \dots, T_n].$$

It is a form of degree m divisible by $T_1 \cdots T_n$. Moreover, for any $s_1, \dots, s_n \in K$,

$$f(s_1, \dots, s_n) = g\left(\sum_{(i)} s_1^{i_1} \cdots s_n^{i_n} e_1^{(i_1)} \cdots e_n^{(i_n)} \right) = 0,$$

$$\frac{\partial f}{\partial T_k}(s_1, \dots, s_n) = g\left(\sum_{(i)} i_k s_1^{i_1} \cdots s_k^{i_k-1} \cdots s_n^{i_n} e_1^{(i_1)} \cdots e_n^{(i_n)} \right) = 0,$$

and hence $f = 0$ by (4). Consequently $g = 0$, as desired.

(5) $\Rightarrow$ (6). Let $R' = R[T]/(T^2) = R \oplus Rt$ where $t^2 = 0$. It follows from [5, Corollary 6.2] that $\tilde{\Gamma}^{m,n}(R')/t\tilde{\Gamma}^{m,n}(R') \simeq \tilde{\Gamma}^{m,n}(R'/tR') = \tilde{\Gamma}^{m,n}(R)$. Since R is a retract of R' we obtain that $\tilde{\Gamma}^{m,n}(R') = \tilde{\Gamma}^{m,n}(R) \oplus t\tilde{\Gamma}^{m,n}(R')$ as R-modules. Consequently, $M\tilde{\Gamma}^{m,n}(R') = M\tilde{\Gamma}^{m,n}(R) \oplus Mt\tilde{\Gamma}^{m,n}(R')$, and hence $t\tilde{\Gamma}^{m,n}(R') \cap M\tilde{\Gamma}^{m,n}(R') = Mt\tilde{\Gamma}^{m,n}(R')$. Since $t(\tilde{\Gamma}^{m,n}(R')/M\tilde{\Gamma}^{m,n}(R')) = t\tilde{\Gamma}^{m,n}(R'/MR') = t\tilde{\Gamma}^{m,n}(K[T]/(T^2)) = 0$ by (5), we obtain that $t\tilde{\Gamma}^{m,n}(R') \subset M\tilde{\Gamma}^{m,n}(R')$, and therefore $t\tilde{\Gamma}^{m,n}(R') = Mt\tilde{\Gamma}^{m,n}(R')$. Observe that $t\tilde{\Gamma}^{m,n}(R')$ is a finitely generated R-module, so $t\tilde{\Gamma}^{m,n}(R') = 0$ by the Nakayama Lemma.

(6) $\Rightarrow$ (1). Let $t \in M$. Substituting T by t we obtain an epimorphism

$$\tilde{\Gamma}^{m,n}(R[T]/(T^2)) \to \tilde{\Gamma}^{m,n}(R/(t^2)) \simeq \tilde{\Gamma}^{m,n}(R)/t^2\tilde{\Gamma}^{m,n}(R).$$

Then (6) gives us $t\tilde{\Gamma}^{m,n}(R) \subset t^2\tilde{\Gamma}^{m,n}(R) \subset M(t\tilde{\Gamma}^{m,n}(R))$. By the Nakayama Lemma, $t\tilde{\Gamma}^{m,n}(R) = 0$. □

6. The Main Theorem

In this section, we find all natural n for which Hom^m is n-strong, i.e., such that Hom^m admits a system of strong n-covering relations.

Following [5], we define $d(R) = \min\{|R/M| : M \in \mathrm{Max}(R)\}$ for any (commutative) ring R. It is a natural number or ∞. We need the following:

Lemma 6.1. *If R is an S-algebra, then $d(S) \le d(R)$. Moreover,*

$$d(S) = \min\{d(S_M) : M \in \mathrm{Max}(S)\}.$$

Proof. Suppose that $M \in \mathrm{Max}(R)$ and R/M is finite. Then $f : S \to R$ induces a monomorphism $S/f^{-1}(M) \hookrightarrow R/M$, and hence $S/f^{-1}(M)$ is a finite field with $|S/f^{-1}(M)| \le |R/M|$. Consequently, $d(S) \le d(R)$. The second part is obvious. □

We are ready to prove

Main Theorem 6.2. *Let S be a (commutative) ring, $d = d(S)$ and m, $n > 0$. The following statements are equivalent*:

(1) Hom^m *is n-strong over S*;

(2) $\mathrm{Ker}(h^{m,n}(R)) \to \mathrm{Ker}(h^{m,n}(R/I))$ *is an epimorphism for each S-algebra R and each ideal $I \subset R$*;

(3) $M\tilde{\Gamma}^{m,n}(R) = 0$ *for each local S-algebra (R, M)*;

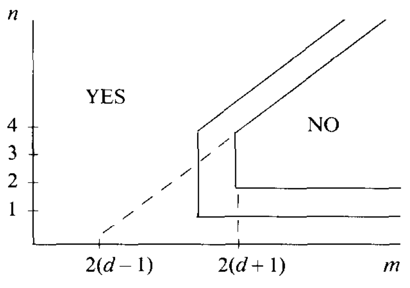
(4) $M\tilde{\Gamma}^{m,n}(R) = 0$ *for some local S-algebra (R, M) satisfying $|R/M| = d$ and $M \ne M^2$*;

(5) $T\tilde{\Gamma}^{m,n}(K[T]/(T^2)) = 0$ *for every quotient field K of S*;

(6) $T\tilde{\Gamma}^{m,n}(S[T]/(T^2)) = 0$;

(7) $n = 1$ *or* $m < 2(d+1)$ *or* $m - n < 2(d-1)$.

The following figure shows when the above conditions are satisfied:



In particular, Hom^m *is strong over S iff $m < 2(d+1)$.*

Proof. First of all, observe that (3), (4) and (5) are equivalent to (7) by Lemma 6.1 and Theorem 5.1 (being also true for $q = \infty$ because $\tilde{\Gamma}^{m,n}(R) = 0$ for $d(R) = \infty$, see [5]). Similarly, (6) $\Leftrightarrow$ (7) by localization.

(3) $\Rightarrow$ (2). Let us consider the following commutative diagram with exact rows over the ring R:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}(h^{m,n}(R)) \arrow[r] \arrow[d, "c"] & \Delta^{m,n}(R) \arrow[r, "h^{m,n}(R)"] \arrow[d, "d"] & \tilde{\Gamma}^{m,n}(R) \arrow[r] \arrow[d, "e"] & 0 \\
0 \arrow[r] & \mathrm{Ker}(h^{m,n}(R/I)) \arrow[r] & \Delta^{m,n}(R/I) \arrow[r, "h^{m,n}(R/I)"] & \tilde{\Gamma}^{m,n}(R/I) \arrow[r] & 0
\end{tikzcd}$$

where $d((\Delta^n\delta^m)(r_1e_1, \dots, r_ne_n)) = (\Delta^n\delta^m)(\bar{r}_1\bar{e}_1, \dots, \bar{r}_n\bar{e}_n)$ and $e((\Delta^n\gamma^m)(r_1e_1, \dots, r_ne_n)) = (\Delta^n\gamma^m)(\bar{r}_1\bar{e}_1, \dots, \bar{r}_n\bar{e}_n)$ ($\gamma^m(x)$ denotes $x^{(m)}$). By the Snake Lemma, it suffices to check that $h:\mathrm{Ker}(d)\to\mathrm{Ker}(e)$ is an epimorphism. This can be proved locally, and hence we can assume that R is a local S-algebra. In fact, $\Delta^{m,n}$, $\bar{\Gamma}^{m,n}$ and $h^{m,n}$ commute with localizations (see [7]). Moreover, if $M\in\mathrm{Max}(R)$, then $\Delta^{m,n}(R/I)\otimes_R R_M=\Delta^{m,n}(R/I)\otimes_{R/I}R_M/IR_M=\Delta^{m,n}(R_M/IR_M)$ since R_M/IR_M is the localization of R/I at M/I (the case of $I\not\subset M$ is evident). Similarly, $\bar{\Gamma}^{m,n}(R/I)\otimes_R R_M=\bar{\Gamma}^{m,n}(R_M/IR_M)$, and $h^{m,n}(R/I)$ localizes to $h^{m,n}(R_M/IR_M)$.

Let now (R,M) be a local S-algebra. We can assume that $I\subset M$ since the case $I=R$ is obvious. Then (3) gives us $I\Gamma^{m,n}(R)\subset\bar{\Gamma}^{m,n}(R)$. On the other hand, Γ^m commutes with any change of the base ring (see [9]), and hence $\mathrm{Ker}(\Gamma^{m,n}(R)\to\Gamma^{m,n}(R/I))=I\Gamma^{m,n}(R)$. Consequently, $\mathrm{Ker}(e)=\bar{\Gamma}^{m,n}(R)\cap I\Gamma^{m,n}(R)=I\Gamma^{m,n}(R)$. It follows from [7, Corollary 3.10] that $\mathrm{Ker}(d)=I\Delta^{m,n}(R)+N$, where

$$N=R\{(\Delta^n\delta^m)(r_1e_1,\dots,r_ne_n)-(\Delta^n\delta^m)(s_1e_1,\dots,s_ne_n);\ r_i-s_i\in I,\ i=1,\dots,n\}.$$

Our goal is to check that $I\Gamma^{m,n}(R)\subset(h^{m,n}(R))(I\Delta^{m,n}(R)+N)$.

It follows from (3) that $\tilde{\Gamma}^{m,n}(R)=\tilde{\Gamma}^{m,n}(R)/M\tilde{\Gamma}^{m,n}(R)=\tilde{\Gamma}^{m,n}(K)$ for $K=R/M$, therefore we have the following commutative diagram:

$$\begin{array}{ccccc}
\Gamma^{m,n}(R) & \xrightarrow{\ v\ } & \tilde{\Gamma}^{m,n}(R) & \longrightarrow & 0\\
\downarrow & & \| & & \\
\Gamma^{m,n}(R)/M\Gamma^{m,n}(R)=\Gamma^{m,n}(K) & \longrightarrow & \tilde{\Gamma}^{m,n}(K) & \longrightarrow & 0
\end{array}$$

We can treat $\tilde{\Gamma}^{m,n}(K)$ as a subspace of $\Gamma^{m,n}(K)$ and hence there exist elements $E_1,\dots,E_t\in\Gamma^{m,n}(R)$ such that $\bar{E}_1,\dots,\bar{E}_t$ form a basis of $\Gamma^{m,n}(K)$ and $\bar{E}_1,\dots,\bar{E}_s$ (for some $s\le t$) form a basis of $\tilde{\Gamma}^{m,n}(K)$. Obviously, $E_1,\dots,E_t$ form a minimal set of generators of $\Gamma^{m,n}(R)$. Since $\Gamma^{m,n}(R)$ is free (see [9]) it follows that $\{E_1,\dots,E_t\}$ is a basis of $\Gamma^{m,n}(R)$. Consequently,

$$\bar{\Gamma}^{m,n}(R)=\mathrm{Ker}(v)=ME_1\oplus\cdots\oplus ME_s\oplus RE_{s+1}\oplus\cdots\oplus RE_t. \tag{6.1}$$

Suppose that the standard basis of $\Gamma^{m,n}(R)$ has the following presentation:

$$e_1^{(i_1)}\cdots e_n^{(i_n)}=\sum_{i=1}^{t}a_i^{i_1,\dots,i_n}E_i,\quad a_i^{i_1,\dots,i_n}\in R,\ i_1+\cdots+i_n=m,\ i_1,\dots,i_n>0,$$

and consider the polynomials

$$f_i=\sum_{(i)}a_i^{i_1,\dots,i_n}T_1^{i_1}\cdots T_n^{i_n}\in R[T_1,\dots,T_n],\quad i=1,\dots,t,$$

where $(i)=(i_1,\dots,i_n)$, $i_1+\cdots+i_n=m$, $i_1,\dots,i_n>0$. Observe that

$$\begin{aligned}(h^{m,n}(R))((\Delta^n\delta^m)(r_1e_1,\dots,r_ne_n))&=(\Delta^n\gamma^m)(r_1e_1,\dots,r_ne_n)\\
&=\sum_{(i)}r_1^{i_1}\cdots r_n^{i_n}e_1^{(i_1)}\cdots e_n^{(i_n)}=\sum_{i=1}^{t}\Big(\sum_{(i)}a_i^{i_1,\dots,i_n}r_1^{i_1}\cdots r_n^{i_n}\Big)E_i\end{aligned} \tag{6.2}$$

$$= \sum_{i=1}^{t} f_i(r_1, \dots, r_n) E_i.$$

Let us assume for a moment that $R = K[[T]]$. Then $\overline{a_i^{i_1, \dots, i_n}} \in K \subset R$ and we can lift $\bar{E}_1, \dots, \bar{E}_t \in \Gamma^{m,n}(K)$ to a basis $\{E_1, \dots, E_t\}$ of $\Gamma^{m,n}(R)$ satisfying

$$e_1^{(i_1)} \cdots e_n^{(i_n)} = \sum_{i=1}^{t} \overline{a_i^{i_1, \dots, i_n}} E_i.$$

Hence $\bar{\Gamma}^{m,n}(R)$ is generated by elements

$$(h^{m,n}(R))((\Delta^n \delta^m)(r_1 e_1, \dots, r_n e_n)) = \sum_{i=1}^{t} \bar{f}_i(r_1, \dots, r_n) E_i, \quad r_1, \dots, r_n \in R,$$

where $\bar{f}_i = \sum_{(i)} \overline{a_i^{i_1, \dots, i_n}} T_1^{i_1} \cdots T_n^{i_n} \in K[T_1, \dots, T_n] \subset R[T_1, \dots, T_n]$. Therefore, by (6.1), the submodule $\Gamma = \bigoplus_{i=1}^{s} ME_i \subset \Gamma^{m,n}(R) = \bigoplus_{i=1}^{t} RE_i$ is generated by elements

$$\sum_{i=1}^{s} \bar{f}_i(r_1, \dots, r_n) E_i, \quad r_1, \dots, r_n \in R.$$

Since $R = K[[T]]$ is a principal ideal domain and the submodule Γ has elementary divisors

$$(\underbrace{T, \dots, T}_{s}, \underbrace{0, \dots, 0}_{t-s}),$$

it follows from [1, §4, Proposition 3] that T^s is the g.c.d. of $s \times s$ minors of the matrix $(\bar{f}_i(r_1, \dots, r_n))$ defined for $i = 1, \dots, s$ and $(r_1, \dots, r_n) \in R^n$. Hence T^s is associated with a minor $\det(\bar{f}_i(g_{1j}, \dots, g_{nj}))_{i,j=1,\dots,s}$ for some $g_{kj} \in K[[T]]$.

Let $g_{kj} \equiv \bar{x}_{kj} + T\bar{y}_{kj} \pmod{T^2}$ for some $\bar{x}_{kj}, \bar{y}_{kj} \in K$. Since $\bar{f}_i(\bar{x}_{1j}, \dots, \bar{x}_{nj}) \in K \cap M = 0$ for $i \le s$, the Taylor expansion gives us

$$\bar{f}_i(g_{1j}, \dots, g_{nj}) \equiv T \sum_{k=1}^{n} \bar{y}_{kj} \frac{\partial \bar{f}_i}{\partial T_k}(\bar{x}_{1j}, \dots, \bar{x}_{nj}) \pmod{T^2}$$

for $i, j = 1, \dots, s$. Consequently,

$$\det(\bar{f}_i(g_{1j}, \dots, g_{nj})) \equiv T^s \det\left(\sum_{k=1}^{n} \bar{y}_{kj} \frac{\partial \bar{f}_i}{\partial T_k}(\bar{x}_{1j}, \dots, \bar{x}_{nj}) \right) \pmod{T^{s+1}},$$

and hence the determinant on the right-hand side is a non-zero element of K. Since

$$\det\left(\sum_{k=1}^{n} \bar{y}_{kj} \frac{\partial \bar{f}_i}{\partial T_k}(\bar{x}_{1j}, \dots, \bar{x}_{nj}) \right)$$
$$= \sum_{k_1=1}^{n} \cdots \sum_{k_s=1}^{n} \bar{y}_{k_1 1} \cdots \bar{y}_{k_s 1} \det\left(\frac{\partial \bar{f}_i}{\partial T_{k_j}}(\bar{x}_{1j}, \dots, \bar{x}_{nj}) \right),$$

it follows that

$$\det\left(\frac{\partial \bar{f}_i}{\partial T_{k_j}}(\bar{x}_{1j}, \dots, \bar{x}_{nj}) \right) \neq 0 \tag{6.3}$$

for some $k_1,\dots,k_s\in\{1,\dots,n\}$ and some $\bar{x}_{kj}\in K$, $k=1,\dots,n$, $j=1,\dots,s$.

Returning to the general case we prove that $IE_p\subset(h^{m,n}(R))(I\Delta^{m,n}(R)+N)=\mathrm{Im}(h)$ for $p=1,\dots,t$. If $i>s$, then $E_p\in\bar{\Gamma}^{m,n}(R)=\mathrm{Im}(h^{m,n}(R))$ and hence $IE_p\subset(h^{m,n}(R))(I\Delta^{m,n}(R))$. Let $p\le s$. It follows from (6.3) that the matrix

$$A=\left(\frac{\partial f_i}{\partial T_{k_j}}(x_{1j},\dots,x_{nj})\right)_{i,j=1,\dots,s}\qquad (x_{kj}\in R)$$

is invertible in R, and therefore any vector of R^s is of the form $(r_1,\dots,r_s)A^{\mathrm{t}}$ for some $r_1,\dots,r_s\in R$. Consequently,

$$E_p=\sum_{i=1}^{s}\left(\sum_{j=1}^{s} r_j\frac{\partial f_i}{\partial T_{k_j}}(x_{1j},\dots,x_{nj})\right)E_i.$$

Observe that

$$f_i(r_1,\dots,r_k+u,\dots,r_n)-f_i(r_1,\dots,r_n)\equiv u\frac{\partial f_i}{\partial T_k}(r_1,\dots,r_n)\pmod{u^2}$$

for any $u\in I$. Denoting by $\equiv$ the congruence modulo $u^2\Gamma^{m,n}(R)+(h^{m,n}(R))(I\Delta^{m,n}(R)+N)$, we compute

$$\begin{aligned}
uE_p&=\sum_{j=1}^{s} r_j\left(\sum_{i=1}^{s} u\frac{\partial f_i}{\partial T_{k_j}}(x_{1j},\dots,x_{nj})E_i\right)\\
&\equiv\sum_{j=1}^{s} r_j\left(\sum_{i=1}^{s}(f_i(x_{1j},\dots,x_{k_jj}+u,\dots,x_{nj})-f_i(x_{1j},\dots,x_{nj}))E_i\right)\\
&\equiv\sum_{j=1}^{s} r_j\left(\sum_{i=1}^{t}(f_i(x_{1j},\dots,x_{k_jj}+u,\dots,x_{nj})-f_i(x_{1j},\dots,x_{nj}))E_i\right)\\
&=\sum_{j=1}^{s} r_j(h^{m,n}(R))((\Delta^n\delta^m)(x_{1j}e_1,\dots,(x_{k_jj}+u)e_{k_j},\dots,x_{nj}e_n)\\
&\qquad\qquad -(\Delta^n\delta^m)(x_{1j}e_1,\dots,x_{nj}e_n))
\end{aligned}$$

by formula (6.2). Since $u\in I$ it follows that

$$uE_p\in u^2\Gamma^{m,n}(R)+(h^{m,n}(R))(I\Delta^{m,n}(R)+N).$$

Hence

$$u\Gamma^{m,n}(R)\subset M(u\Gamma^{m,n}(R))+(u\Gamma^{m,n}(R)\cap\mathrm{Im}(h))\quad\text{for any } u\in I.$$

By the Nakayama Lemma $u\Gamma^{m,n}(R)\subset\mathrm{Im}(h)$, and finally $I\Gamma^{m,n}(R)\subset\mathrm{Im}(h)$, as desired.

(2) $\Rightarrow$ (1). Let R be an S-algebra and let I be an ideal in R. By Proposition 4.5 we must prove that $K^n(R)\to K^n(R/I)$ is an epimorphism for $K=K_{\mathrm{Hom}^m}$. Let us consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Ker}(b) \arrow[r, "p"] \arrow[d] & \mathrm{Ker}(d) \arrow[r, "h"] \arrow[d] & \mathrm{Ker}(e) \arrow[d] \\
F^n(R) \arrow[r] \arrow[d, "b"] & \Delta^{m,n}(R) \arrow[r, "h^{m,n}(R)"] \arrow[d, "d"] & \bar{\Gamma}^{m,n}(R) \arrow[d, "e"] \\
F^n(R/I) \arrow[r] & \Delta^{m,n}(R/I) \arrow[r, "h^{m,n}(R/I)"] & \bar{\Gamma}^{m,n}(R/I)
\end{tikzcd}$$

Since $K^n = \mathrm{Ker}(F^n \to \bar{\Gamma}^{m,n})$ we must prove (by the Snake Lemma) that hp is an epimorphism. Using again the Snake Lemma we conclude from (2) that h is an epimorphism and, moreover, that p is an epimorphism since Appl^m is a strong ED-functor.

(1) $\Rightarrow$ (5). Let $R = K[[T]]$ and $M = (T)$. Since R is a discrete valuation ring and $\bar{\Gamma}^{m,n}(R)$ is a torsion R-module (see [5, Corollary 7.1] or [7, Corollary 5.8]) it follows that $\bar{\Gamma}^{m,n}(R)$ is a direct sum of R-modules of the form R/M^s. We prove that all these s are equal to 1. Then $T\bar{\Gamma}^{m,n}(R) = 0$ and consequently $T\bar{\Gamma}^{m,n}(K[T]/(T^2)) = T\bar{\Gamma}^{m,n}(R/M^2) = 0$, as desired.

Suppose that $\bar{\Gamma}^{m,n}(R)$ contains a direct summand isomorphic to R/M^s for some $s \geq 2$. There exists a basis $\{E_1, \ldots, E_t\}$ of $\Gamma^{m,n}(R)$ such that $M^s E_1$ is a direct summand of $\bar{\Gamma}^{m,n}(R)$. Let $\bar{R} = R/M^s$. Since $\Gamma^{m,n}(\bar{R}) = \Gamma^{m,n}(R) \otimes_R \bar{R}$ and $\bar{\Gamma}^{m,n}(\bar{R}) = \bar{\Gamma}^{m,n}(R) \otimes_R \bar{R}$ (see [9] and [7], respectively), it follows that $\bar{R}\bar{E}_1 \simeq \bar{R}$ is a direct summand of $\Gamma^{m,n}(\bar{R})$ and $\bar{R}/\bar{M}^s = \bar{R}$ is the suitable direct summand of $\bar{\Gamma}^{m,n}(\bar{R})$. Consequently, $\bar{R}\bar{E}_1 \cap \bar{\Gamma}^{m,n}(\bar{R}) = 0$, and therefore $M^s E_1$ is contained in $\mathrm{Ker}(e : \bar{\Gamma}^{m,n}(R) \to \bar{\Gamma}^{m,n}(\bar{R}))$ (as a direct summand).

Consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Ker}(b) \arrow[r] \arrow[d] & \mathrm{Ker}(e) \arrow[r, "\Pi'"] \arrow[d] & M^s \arrow[d, equal] \\
g : F^n(R) \arrow[r] \arrow[d, "b"] & \bar{\Gamma}^{m,n}(R) \arrow[r, "\bar{\Pi}"] \arrow[d, "e"] & M^s \arrow[d] \\
F^n(\bar{R}) \arrow[r] & \bar{\Gamma}^{m,n}(\bar{R}) \arrow[r] & 0
\end{tikzcd}$$

where the splitting epimorphisms $\bar{\Pi}$ and Π' are restrictions of the projection Π: $\Gamma^{m,n}(R) \to R$, $\Pi(E_i) = \delta_{i1}$. Observe that $\mathrm{Ker}(b) \to \mathrm{Ker}(e)$ is an epimorphism by (1) and the Snake Lemma, and hence the upper composition is also an epimorphism.

In other words, $g(\mathrm{Ker}(b)) = M^s$.

Observe that

$$\bar{\Pi}((\Delta^n \gamma^m)(r_1 e_1, \dots, r_n e_n)) = \bar{\Pi}\Big(\sum_{(i)} r_1^{i_1} \cdots r_n^{i_n} e_1^{(i_1)} \cdots e_n^{(i_n)} \Big)$$

$$= \sum_{(i)} \Pi(e_1^{(i_1)} \cdots e_n^{(i_n)}) r_1^{i_1} \cdots r_n^{i_n} = f(r_1, \dots, r_n)$$

for some polynomial $f \in R[T_1, \dots, T_n]$, where (i) runs over systems $(i_1, \dots, i_n)$ satisfying $i_1 + \cdots + i_n = m$ and $i_1, \dots, i_n > 0$. Therefore g is defined as follows:

$$g((r_1 e_1, \dots, r_n e_n)) = \bar{\Pi}((\Delta^n \gamma^m)(r_1 e_1, \dots, r_n e_n)) = f(r_1, \dots, r_n).$$

It is easy to see that, for any set Σ, $\mathrm{Ker}(F_R(\Sigma) \to F_{R/I}(\Sigma)) = IF_R(\Sigma)$ and $\mathrm{Ker}(F_{\bar{R}}(\Sigma) \to F_{\bar{R}}(\Sigma/\sim)) = \bar{R}\{[x] - [y] : x \sim y\}$. Consequently,

$$\mathrm{Ker}(b) = M^s F^n(R)$$
$$+ R\{((r_1 + t_1)e_1, \dots, (r_n + t_n)e_n) - (r_1 e_1, \dots, r_n e_n) : r_i \in R,\ t_i \in M^s\}.$$

We prove that $g(\mathrm{Ker}(b)) \subset M^{s+1} \not\supseteq M^s$, which will give us a contradiction. First of all, $g(M^s F^n(R)) \subset M^s g(F^n(R)) \subset M^{2s} \subset M^{s+1}$. Let now $r_i, t_i \in R$, $i = 1, \dots, n$. Then

$$g(((r_1 + t_1)e_1, \dots, (r_n + t_n)e_n) - (r_1 e_1, \dots, r_n e_n))$$
$$= f(r_1 + t_1, \dots, r_n + t_n) - f(r_1, \dots, r_n)$$
$$\equiv \sum_{i=1}^{n} t_i \frac{\partial f}{\partial T_i}(r_1, \dots, r_n) \mod (t_1, \dots, t_n)^2,$$

and the left-hand side belongs to M^s. Taking $t_j = \delta_{ij} T^{s-1}$, we get $T^{s-1}(\partial f / \partial T_i)(r_1, \dots, r_n) \in M^s$ since $2(s-1) \geq s$. Consequently, $(\partial f / \partial T_i)(r_1, \dots, r_n) \in M$ for any $r_1, \dots, r_n \in R$. Finally, taking $t_1, \dots, t_n \in M^s$ we obtain that

$$g(((r_1 + t_1)e_1, \dots, (r_n + t_n)e_n) - (r_1 e_1, \dots, r_n e_n)) \in M^{s+1}$$

as desired. This completes the proof. $\square$

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